Blowing-up points on l.c.K. manifolds.

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Abstract

It is a classical result, due to F. Tricceri, that the blow-up of a manifold of locally conformally Kähler (l.c.K. for short) type at some point is again of l.c.K. type. However, the proof given in [5] is somehow unclear. We give a different argument to prove the result, using "standard tricks" in algebraic geometry.

Keywords: Blow-up of a manifold at a point, locally conformally Kähler manifold, Lee form.
2000 Mathematics subject classification: 53C55, 14E99

1 Introduction

We begin by recalling the basic definitions and facts; details can be found for instance in the book [2].

Definition 1 *Let (X, J) be a complex manifold. A hermitian metric g on it is called* locally conformally Kähler*, l.c. K. for short, if there exists some open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of X such that for each $\alpha \in A$ there is some smooth function f_α defined on U_α such that the metric $g_\alpha = e^{-f_\alpha} g$ is Kähler.*

A complex manifold (X, J) will be called of l.c.K. type if it admits an l.c.K. metric

Letting ω to be the *Kähler form* associated to g by $\omega(X, Y) = g(X, JY)$, one can immediately show that the above definition is equivalent to the existence of a closed $1-$form θ such that $d\omega = \theta \wedge \omega$. The form θ is called *the Lee form* of the metric g. It is almost immediate to see that θ is closed; iy is exact iff the metric g is global conformally equivalent to a Kähler metric. Usually, by an l.c.K. manifold one understands a hermitian manifold whose metric is

not globally conformally Kähler. In particular, the first Betti number of an l.c.K. manifold is always strictly positive; more, for compact Vaisman manifolds (l.c.K. with parallel Lee form) the fundamental group fits into an exact sequence

$$0 \to G \to \pi_1(M) \to \pi_1(X) \to 0$$

where $\pi_1(X)$ is a fundamental group of a Kähler orbifold, and G a quotient of $\mathbb{Z}^2$ by a subgroup of rank ≥ 1 (see [4]). Moreover, the l.c.K. class is not stable to small deformations: some Inoue surfaces do not admit l.c.K. structures and they are complex deformations of other Inoue surfaces with l.c.K. metrics (see [5], [1]).

However, l.c.K. manifolds share with the Kähler ones the property of being closed under blowing-up points. To can state the result, let X be a a complex manifold and $P \in X$ some fixed point. We denote by $\widehat{X}$ the manifold obtained by blowing-up P, by $c : \widehat{X} \to X$ the blowing-up map and E the exceptional divisor of π (i.e. $E = c^{-1}(\{P\})$). The goal is to prove the following

Theorem 1 *If the complex manifold X carries an l.c.K. metric, then so does its blow-up $\widehat{X}$ at any point.*

The result was stated in [5], but the proof in this paper has a gap.

For the sake of completeness, we include in the next section some basic facts about blow-up's of points on complex manifolds. Eventually, in the last section we prove the theorem.

2 Basic facts about blow-up's of points.

This section is entirely standard and is almost an verbatim reproduction of facts from classical texts, as for instance [3].

Let X be a complex, $n-$dimensional manifold. Let $P \in X$ be a point; choose a holomorphic local coordinate system $(x_1, \dots, x_n)$ defined in some open neighborhood U of P such that $x_1(P) = \cdots = x_n(P) = 0$. Consider the manifold $U \times \mathbb{P}^{n-1}(\mathbb{C})$ and assume $[y_1 : \cdots : y_n]$ is some fixed homogenous coordinate system on $\mathbb{P}^{n-1}(\mathbb{C})$. Let $\widehat{U} \subset U \times \mathbb{P}^{n-1}(\mathbb{C})$ be the closed subset defined by the system of equations $x_i y_j = x_j y_i, 1 \leq i < j \leq n$. One can check that $\widehat{U}$ is actually a submanifold of $U \times \mathbb{P}^{n-1}(\mathbb{C})$. Moreover, the restriction of the projection onto the first factor $c : \widehat{U} \to U$ has the following properties: the

fiber of c above P, $c^{-1}(\{P\})$, is a submanifold E of $\widehat{U}$ which is biholomorphic to $\mathbb{P}^{n-1}(\mathbb{C})$ and the restriction of c at $\widehat{U}\setminus E$ defines a biholomorphism between $\widehat{U}\setminus E$ and $U\setminus\{P\}$. Using it, we can glue $\widehat{U}$ to X along $U\setminus\{P\}$. The resulting manifold will usually be denoted by $\widehat{X}$; the map c above extends obviously to a map - denoted by the same letter- $c:\widehat{X}\to X$. Notice that on one hand c is a biholomorphic map between $\widehat{X}\setminus E$ and $X\setminus\{P\}$ and, on the other hand, c "contracts" E, i.e. $c(E)=\{P\}$ (E is called accordingly the "exceptional divisor" of c).

Let now $y\in\widehat{X}$ be some point. If $y\notin E$, then the tangent map

$$c_{*,y}:T_y(\widehat{X})\to T_{c(y)}(X)$$

is a isomorphism, while if $y\in E$ then the rank of this map is one and its kernel consists of those vectors that are tangent at y to E, i.e. $Ker(c_{*,y})=T_y(E)$.

Next, recall that to each closed complex submanifold E of codimension one of some complex manifold X one can associate a holomorphic vector bundle, usually denoted $\mathcal{O}_X(E)$; see e.g. [3], Chapter 1, Section 1. If one chooses a hermitian metric h in $\mathcal{O}_X(E)$ there exists and is unique a linear connection D in the vector bundle which is also compatible with the complex structure (see e.g. the Lemma on page 73, [3]). The curvature Ω_E of this connection is a closed $(1,1)-$form.

We shall next exemplify the computation of the curvature of a metric connection in the special case we are interested in, namely when E is the exceptional divisor of some blow-up. So let X be a manifold, $P\in X$, U a coordinate neighborhood of P as in the beginning of the section and $\widehat{X}$ the blow-up of X at P. For ε small enough set

$$U_{2\varepsilon}\overset{def}{=}Q\in U|\ |x_i(Q)|<2e \text{ for all } i=1,\ldots,n\}.$$

Let $\pi':U\times\mathbb{P}^{n-1}(\mathbb{C})\to\mathbb{P}^{n-1}(\mathbb{C})$ be the projection onto the the second factor; then $\mathcal{O}_{\widehat{U}}(E)=\pi'^{*}(\mathcal{O}_{\mathbb{P}^{n-1}(C)}(-1))$. Let ω_{FS} be the Kähler form of the Fubini-Study metric on $\mathbb{P}^{n-1}(\mathbb{C})$; then $-\omega_{FS}$ is the curvature of the canonical connection of the natural metric h in the tautological line bundle $\mathcal{O}_{\mathbb{P}^{n-1}(C)}(-1)$. Let $h'\overset{def}{=}\pi'^{*}(h)$ be the induced metric in $\mathcal{O}_{\widehat{U}}(E)$; then its curvature will be $\pi'^{*}(-\omega_{FS})$. On the other hand, the line bundle $\mathcal{O}_{\widehat{X}}(E)$ is trivial outside E; fix a nowhere vanishing section σ of it and let h" be the unique metric making σ into a unitary basis. Let now ϱ_1,ϱ_2 be a partition of unity such that $\varrho_1\equiv 1$ on U_ε and $\varrho_1\equiv 0$ outside $U_{2\varepsilon}$ and respectively $\varrho_2\equiv 0$

on U_ε and $\equiv 1$ outside $U_{2\varepsilon}$. Let $h = \varrho_1 h' + \varrho_2 h"$; it is a hermitian metric on $\mathcal{O}_X(E)$. Its curvature will be zero outside $U_{2\varepsilon}$ since $h = h"$ there. In U_ε, its curvature will be the pull-back (via π') of $-\omega_{FS}$, hence it is semi-negative definite; moreover, its restriction to E will be negative definite on vectors that are tangent along E, since the restriction of π' to E is a biholomorphism between E and $\mathbb{P}^{n-1}(\mathbb{C})$..

3 Proof of the theorem.

Proof. First, let us fix the terminology. We will say that a $(1,1)-$form ω on a complex manifold (M, J_M) is positive (semi-)definite if for any point $m \in M$ and any non-zero tangent vector $v \in T_m M$ one has $\omega(v, J_M v) > 0$ (respectively ≥ 0), in other words if it is the Kähler form of some hermitian metric on M.

Let now ω be the Kähler form of an l.c.K. metric on X. We see $c^*(\omega)$ is a $(1,1)-$ form on $\widehat{X}$ which is positive definite on $X \setminus E$ and satisfies $dc^*(\omega) = c^*(\theta) \wedge c^*(\omega)$, where θ is the Lee form of the given l.c.K. metric on X. As E is simply connected we see (e.g. by using Lemma 4.4 in [5]) there exists an open neighborhood U of E and a smooth function $f : \widehat{X} \to \mathbb{R}$ such that $\omega' \overset{def}{=} e^f c^*(\omega)$ satisfies $d\omega' = \theta' \wedge \omega'$ and such that $\theta'_{|U} \equiv 0$.

On the other hand, we can find a hermitian metric in the holomorphic line bundle $\mathcal{O}_{\widehat{X}}(E)$ on $\widehat{X}$ associated to E such that the curvature Ω_E of its canonical connection is negative definite along E (i.e. $\Omega_E(v, J_{\widehat{X}} v) < 0$ for every non-vanishing vector $v \in T_P(E)$ and for every $P \in E$), is negatively semidefinite at points of E (i.e. $\Omega_E(v, J_{\widehat{X}} v) \leq 0$ for any $P \in E$ and any $v \in T_P(\widehat{X})$) and is zero outside U (cf. e.g. [3], pp 185-187). Notice that Ω_E is closed.

We infer that for some positive integer N the $(1,1)-$form $h \overset{def}{=} N\omega' - \Omega_E$ is positive definite.

Indeed, this is obvious outside U as Ω_E vanishes here and $N\omega'$ is positive definite for any $N > 0$.

Along E, as both ω' and $-\Omega_E$ are positive semidefinite, we have only to check the definiteness of h. Let $y \in E$ be some point and $v \in T_y(\widehat{X})$. Assume $h(v, J_{\widehat{X}} v) = 0$; since both ω' and $-\Omega_E$ are positive semidefinite, we get $\omega'(v, J_{\widehat{X}} v) = \Omega_E(v, J_{\widehat{X}} v) = 0$. But $\omega'(v, J_{\widehat{X}} v) = 0$ implies $c^*(\omega)(v, J_{\widehat{X}} v) = 0$; so $\omega(c_{*,y}(v), J_{\widehat{X}} c_{*,y}(v)) = 0$ hence $v \in Ker(c_{*,y})$. As $Ker(c_{*,y}) = T_y(E)$ we

get that $v \in T_y(E)$; but as $-\Omega_E(v, J_{\widehat{X}} v) = 0$ we see that $v = 0$

To check the assertion on U, it suffices to notice that for each point x in U there exists some n_x such that $N\omega' - \Omega_E$ is positive definite at x for all $N \geq n_x$, hence also positive definite on a neighborhood of x; since U is relatively compact, we can cover it by finitely many such neighborhoods, and take the maximum of the corresponding n_x's.

Last, let us see that $N\omega' - \Omega_E$ is l.c.k. One has

$$d(N\omega' - \Omega_E) = N d\omega' = \theta' \wedge N\omega'.$$

But $\theta' \wedge \Omega_E = 0$ since their supports are disjoint, so we have

$$d(N\omega' - \Omega_E) = \theta' \wedge N\omega' - \theta' \wedge \Omega_E = \theta' \wedge (N\omega' - \Omega_E).$$

Acknowledgments. I wish to thank L.Ornea and I. Vaisman for useful discussions; also, I'm especially grateful to V. Brînzănescu for a careful reading of a preliminary version of this paper.

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